Combinatorial aspects of an autonomous system with ternary collisions and of a three person tournament game

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1. Introduction

In this paper, we shall consider two different kinds of mathematical models which possess a common combinatorial property. One of which is concerned with an invariant of an autonomous system with ternary collisions. While, the other one is a game theoretical model of a generalized "Scissors, paper and Stone game."

Firstly we shall define some terminologies and notations of graphs which will be used throughout this paper. Let $V = \{v_1, \cdots, v_n\}$ and E be a subset of $V \times V$. A pair $T = (V, E)$ is called a *digraph* and the elements of V and E are called *vertices* and *directed edges*, respectively. An $n \times n$ *adjacency matrix* $A = (a_{uv})$ of T is defined by

$$a_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

A digraph T is called a *tournament* if the adjacency matrix A of T satisfies

$$A + A^T + I = J,$$

where I is the $n \times n$ identity matrix, J is the $n \times n$ all-one matrix and A^T is the transpose of A. For any vertices u, v and w of a tournament $T = (V, E)$, let $n_+^v = |\{u \in V | (v, u) \in E\}|$ and $n_-^v = |\{u \in V | (u, v) \in E\}|$. If $n_+^v = n_-^v = const.$ holds for any $v \in V$, then the tournament T is said to be *regular*. For an adjacency matrix A of a tournament T, let

$$H = \begin{pmatrix} 1 & & \cdots & & 1 \\ -1 & & & & \\ \vdots & & A - A^T + I & & \\ -1 & & & & \end{pmatrix}.$$

A tournament T is called an *Hadamard tournament* if $HH^T = H^TH = (n+1)I$ holds.

In Section 2, the condition that an autonomous system has a given invariant is investigated in connection with an Hadamard tournament. In Section 3, similar results to Section 2 are obtained for a "tournament game" which is a generalized "Scissors, Paper and Stone game."

2. An autonomous system with ternary collisions

Assume that n types(species) of particles are moving around a field. And assume that there are dominance relations between every two types. When a collision occurs, the types of the particles may be changed. We correspond each of the n types to a vertex $v \in V$ of a digraph with n verteces and if type u dominates type v then we assume that an directed edge (u, v) exists in E. Let $p_v(t)$ be the amount of the particles of type v at time t. And consider the following autonomous system:

$$\frac{d}{dt}p_u(t) = p_u(t)\left[\delta_1 \sum_{v \in V} p_v(t)\beta_v^u + \delta_2 \sum_{v \in V}\sum_{w \in V} p_v(t)p_w(t)\gamma_{vw}^u\right], \qquad (1)$$

where β_v^u is the increasing rate of the amount of type u when two particles of types u and v make into a collision(*binary collision*) and γ_{vw}^u is the increasing rate of the amount of type u when three particles of types u, v and w make into a collision (*ternary collision*). It is natural to assume that $\gamma_{vw}^u = \gamma_{wv}^u$. Further, we assume the following conditions:

(a) The number of particles are not changed by collisions, that is, $\beta_v^u + \beta_u^v = 0$ and $\gamma_{vw}^u + \gamma_{wu}^v + \gamma_{uv}^w = 0$ hold for any $u, v, w \in V$.

(b) There are no differences between any types of particles except the distinction of types. And the coefficients β_v^u and γ_{vw}^u are determined only by the dominance relations between types u, v and w. Thus, without loss of generality, we can assume that $\beta_v^u = a_{uv} - a_{vu}$ for any u and v. Furthermore, for any $\{u, v, w\}$ and $\{u', v', w'\}$, if the subgraph induced by $\{u, v, w\}$ is isomorphic to that by $\{u', v', w'\}$ and if u, v, w correspond to u', v', w', respectively, then $\gamma_{vw}^u = \gamma_{v'w'}^{u'}$ holds.

Under the conditions (a) and (b), the coefficients γ_{vw}^u are represented as follows by parameters a, b, c, d and e:

(0) $\gamma^u_{vw} = \gamma^u_{wv}$ for any $u, v, w \in V$.

(i) $\gamma^u_{uu} = 0$ for any $u \in V$.

(ii) $\gamma^u_{vv} = 2d$ and $\gamma^v_{uv} = \gamma^v_{vu} = -d$ for any $(u, v) \in E$.

(iii) $\gamma^u_{vv} = -2e$ and $\gamma^v_{uv} = \gamma^v_{vu} = e$ for any $(v, u) \in E$.

(iv) $\gamma^u_{vw} = a$, $\gamma^v_{wu} = b$ and $\gamma^w_{uv} = c$ in the case when (u, v), (u, w), (v, w) $\in E$, where $a + b + c = 0$.

(v) $\gamma^u_{vw} = \gamma^v_{wu} = \gamma^w_{uv} = 0$ in the case when (u, v), (v, w), $(w, u) \in E$.

For any initial values $p_{v_i}(0)$, $(v_i \in V)$, if a fuction $f(p_{v_1}(t), \cdots, p_{v_n}(t))$ is constant not depending on time t, then f is said to be an *invariant* of the system (1). It is easy to show that $\sum_{v \in V} p_v(t)$ is an invariant, since the condition (a) is assumed.

We obtain the following combinatorial properties that the autonomous system (1) has an invariant $\prod_{v \in V} p_v(t)$.

Theorem 1. Let $T = (V, E)$ be an tournament representing the dominance relation of n types of particles. Assume that $\delta_1 > 0$, $\delta_2 = 0$ and $d = e$ in the autonomous system (1). Then $\prod_{v \in v} p_v(t)$ is an invariant if and only if T is a regular tournament.

Theorem 2. Let $T = (V, E)$ be an tournament representing the dominance relation of n types of particles. Assume that $\delta_1 > 0$, $\delta_2 > 0$ and $d = e$ in the autonomous system (1). Then the necessary and sufficient condition that $\prod_{v \in v} p_v(t)$ is an invariant for any parameters a, b, c and $d (= e)$ is that T is an Hadamard tournament.

3. A three person tournament game with conspiracy

In the game "Scissors, Paper and Stone", each player independently chooses either "scissors", "paper", or "stone". If all players choose the same object or if they choose three kinds of objects, then the game is tied. If two kinds of objects are chosen by the players, then players choosing "scissors" beat players choosing "paper"; players choosing "paper" beat players choosing "stone"; and players choosing "stone" beat players choosing "scissors". This game can be generalized as a "tournament game."

In a tournament game, given a tournament T, each player choose a vertex of T independently. And the payoff of each player is defined by the subgraph induced by the vertices chosen by the players.

In this section, we consider three person tournament games in the case when two of the three players conspire with each other to gain more than playing independently. In general, the expected gain may increase by the conspiracy. But we shall show that if we use an Hadamard tournament no effect of conspiracy can occur. Furthermore, we shall show that if there are no effect of conspiracy, then the tournament must be an Hadamard tournament if the class of tournaments is restricted to those having a sharply transitive subgroup.

Before we consider the case of three person tournament game, we shall state some known preliminary results for two person tournament games. In the case of two person tournament games, if both players choose the same vertex, then the game is tied and the payoff is 0. When two players choose vertices u and v, if $(u, v) \in E$ then the player choosing u wins. In this case, the payoff for the winner is 1 and that for loser is -1. Assume that players P_1 and P_2 choose a vertex v with probabilities p_v and q_v, respectively. The probabilities $\boldsymbol{p} = \{p_v\}_{v\in V}$ and $\boldsymbol{q} = \{q_v\}_{v\in V}$ are called *strategies* of P_1 and P_2. The payoff for player P_1 is $\beta_v^u = a_{uv} - a_{vu}$ when players P_1 and P_2 choose vertices u and v, respectively. And the expected payoff for P_1 is

$$E(\boldsymbol{p}, \boldsymbol{q}) = \sum_{u\in V} \sum_{v\in V} \beta_v^u p_u q_v.$$

The strategies $\boldsymbol{p}^* = \{p_v^*\}_{v\in V}$ and $\boldsymbol{q}^* = \{q_v^*\}_{v\in V}$ which satisfy

$$E(\boldsymbol{p}^*, \boldsymbol{q}^*) = \min_{\boldsymbol{q}} \max_{\boldsymbol{p}} E(\boldsymbol{p}, \boldsymbol{q}) = \max_{\boldsymbol{p}} \min_{\boldsymbol{q}} E(\boldsymbol{p}, \boldsymbol{q})$$

are called *optimal* strategies. It is obvious that

$$E(\boldsymbol{p}^*, \boldsymbol{q}^*) = \sum_{u\in V} \sum_{v\in V} \beta_v^u p_u^* q_v^* = 0$$

holds, since the game is "fair." Fisher and Ryan (1991) showed that optimal strategies $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ is determined uniquely. Thus $\boldsymbol{p}^* = \boldsymbol{q}^*$ holds. A tournament T is said to be *positive* if $p_v^* > 0$ for any vertex v.

Now, we shall consider three person tournament games. Three players P_1, P_2 and P_3 choose vertices of a given tournament T. When these three players choose vertices u, v and w, let γ^u_{vw} be the payoff for the player choosing the vertex u. We assume that the payoff γ^u_{vw} satisfies the following conditions:

(c) $\gamma^u_{vw} = \gamma^u_{wv}$ and $\gamma^u_{vw} + \gamma^v_{wu} + \gamma^w_{uv} = 0$ for any $u, v, w \in V$.

(d) For any $\{u, v, w\}$ and $\{u', v', w'\}$, if the subgraph induced by $\{u, v, w\}$ is isomorphic to that by $\{u', v', w'\}$ and if u, v, w correspond to u', v', w', respectively, then $\gamma^u_{vw} = \gamma^{u'}_{v'w'}$ holds. That is, the payoff depends only on the shape of the induced subgraph.

Under the above conditions, γ^u_{vw} must be represented by the same way in (0), (i), (ii), (iii), (iv) and (v) of Section 2.

A three person tournament game is "fair" if three players choose vertices independently, but if two of the three players conspire with each other. They may earn more than playing independently. Assume that players P_1 and P_2 conspire and that they choose vetices u and v with probability p_{uv}. On the other hand, assume that player P_3 chooses vertex w with probability q_w. Then the game can be considered as a two person matrix game between the pair $\{P_1, P_2\}$ and P_3. In this case, the expected payoff for the conspiring pair $\{P_1, P_2\}$ is

$$E = \sum_{u\in V}\sum_{v\in V}\sum_{w\in V.}(\gamma^u_{vw} + \gamma^v_{uw})p_{uv}q_w = -\sum_{u\in V}\sum_{v\in V}\sum_{w\in V}\gamma^w_{uv}p_{uv}q_w$$

and the value of the game, that is, the expected gain for the pair $\{P_1, P_2\}$ with respect to optimal strategies, are given by

$$f_{\{P_1,P_2\}} = \min_{\{q_w\}}\max_{\{p_{uv}\}} E = \max_{\{p_{uv}\}}\min_{\{q_w\}} E.$$

It is obvious that $f_{\{P_1,P_2\}} \geq 0$ holds since the pair $\{P_1, P_2\}$ may earn more than playing independently. If $f_{\{P_1,P_2\}} > 0$ holds, we say that there is an effect of conspiracy, while if $f_{\{P_1,P_2\}} = 0$, we say that there is no effect of conspiracy.

In the following of this paper, we shall consider the combinatorial property for a tournament to have $f_{\{P_1,P_2\}} = 0$, or $f_{\{P_1,P_2\}} > 0$.

Theorem 3. In the case of $e > d$, $f_{\{P_1,P_2\}} > 0$ holds for any positive tournament T and for any a, b and c. That is, in this case, there is an effect of conspiracy for players P_1 and P_2.

Theorem 4. Let T be an Hadamard tournament. Then in the case of $e = d$, $f_{\{P_1,P_2\}} = 0$ holds for any a,b and c. That is, in this case, there are no effect of conspiracy.

Let G be a subgroup of the automorphism group of a tournament $T = (V, E)$. For any two vertices u and v, if there exist exactly one $g \in G$ such that $u^g = v$ holds, then G is said to be *sharply transitive* on V. In this case, we can identify the set of vertices V with the subgroup G.

Theorem 5. Let $T = (V, E)$ be a positive tournament whose automorphism group has a sharply transitive subgroup G on V. Then $f_{\{P_1,P_2\}} > 0$ holds for any a, b, c and for any $e < d$.

Theorem 6. Let $T = (V, E)$ be a positive tournament whose automorphism group contain a sharply transitive subgroup G. In the case of $e = d$, if $f_{\{P_1,P_2\}} = 0$ holds for any a, b and c, then the tournament T is an Hadamard tournament.

Conjecture. Let T be a tournament. If $f_{\{P_1,P_2\}} = 0$ for any parameters a, b and c, then $e = d$ holds and T is an Hadamard tournament.

References

[1] D. C. Fisher and J. Ryan (1991), Tournament games and positive tournaments. *The 22-nd South Eastern Confernce on Combinatorics, Graph Theory and Computing.*